Derivation of the Quadratic Formula

Since it is a polynomial equation, the standard form of a quadratic equation is in descending order: \( y = ax^2 + bx + c \). We have earlier found that in order to find x-intercepts, we set \( y \) equal to zero and solved for \( x \), just as was the case when finding x-intercepts for the lines that come form the linear equations written in the (Standard or General) form, \( Ax + By = C \). Thus we yield: \( 0 = ax^2 + bx + c \), which I prefer to write in it’s reflected form: 

\[ ax^2 + bx + c = 0, \]  

(which just switches the order of the expressions on both sides of the equation.)

Earlier in the semester we solved equations of this form, through the use of factoring and the zero-product property. Quite recently though, we found that not all quadratic equations of the form \( a x^2 + b x + c = 0 \) are factorable, and that in these cases we needed to use a new solution method, that works for all equations of the form \( ax^2 + bx + c = 0 \), that is called “completing the square”, which I abbreviate (CTS). Not all students embrace this new solution method (CTS), but after being exposed to it, prefer to use a third solution method, called the “quadratic formula”, which also works for all equations of the form \( ax^2 + bx + c = 0 \). Quite often students in high school are asked to memorize this formula:

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]

If \( ax^2 + bx + c = 0 \), then \( x = \pm \sqrt{b^2 - 4ac} \). I would like to derive it for you. We will need to use the completing the square (CTS) method and the square root property ( If \( x^2 = b \), then \( x = \pm b \) ), in order to achieve this. Thus we will start with

\[ ax^2 + bx + c = 0, \]

and together we will algebraically manipulate both sides of the equation to yield

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \]
\[
\left( x^2 + \frac{b}{a}x + \_\right) = -\frac{c}{a} + \\
\left( x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} \right) = -\frac{c}{a} + \frac{b^2}{4a^2} \]
\[
\left( x + \frac{b}{2a} \right)^2 = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \]
\[
\left( x + \frac{b}{2a} \right)^2 = \frac{b^2 - 4ac}{4a^2} \]
\[
x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} \]
\[
x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]
\[
x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \]

Thus,
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

QED